Phi 201: Precept 6

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Introduction:[1] In English and other natural languages, basic sentences are composed of noun phrases and verb phrases. Simplest noun phrases are names, like 'Kevin', 'Ryan', and 'Leora'. These correspond to *constant symbols* (e.g. m, n, o,...with or without subscripts) of FOL. Their semantic function is to refer to particular objects in the domain.

More complex noun phrases are formed by combining common nouns with words like, 'every', 'some', 'most', 'the', 'exactly one', 'three', and 'no', giving us noun phrases like 'every student', 'some dog', 'most children', 'the president', 'exactly one student', 'three blind mice', and 'no person in the room'. In our setting, we call these '*quantified expressions*'. They allow us to talk about *quantities of things* or make *general claims*, like 'Someone in the room is a Princeton student', as opposed to talking bout particular things or to make particular claims like the sentence 'Jessie is a Princeton student'.

In FOL, we've reduced these general claims to two *quantifiers*, *everything* (∀) and *something* (∃). This might not seem like much but in addition to expressions that use 'every' or 'some', we are able to define from these two quantifiers expressions like 'every cube', 'three blind mice', 'no tall student', 'whenever', 'exactly one', and 'the president'. We cannot capture some quantified expressions like 'most students', 'many cubes', and 'infinitely many primes'.

Constants and variables We know that constants are like names, referring to particulars, like the name 'Mel' refers to the person Mel. They can appear in a sentence immediately following predicate symbol, like the sentence 'Sm' (Mel is a student). But we would like a way of distinguishing the expression '0+1', which makes use of constants and expresses something particular about the numbers 0 and 1, and 'x+y', which makes use of variables and expresses something general about a class of objects, like the natural numbers.

Variables (e.g. t, u, v, w, x, y, z,....with or with our subscripts) behave like individual constants in that they also follow predicate symbols ('Sx'), but *their semantic function is not to refer to particular objects.* Variables are placeholders that indicate relationships between quantifiers and predicates. Consider the following:

> Sentence with a (monodic, or one-placed) predicate with a constant: *Ph* (Halvorson is a professor).

[1] Much of this discussion can be found in Barker-Plummer, Barwise, and Etchemendy's *Language, Proof, and Logic.* Email for citation.

Sentence with a (dyadic, or two-placed) predicate with constants: Tsb (Sten is taller than Bobby).

Formula with a free variable: Fx (It (x) is F).

This latter formula, however, is NOT a *sentence* of FOL. To better understand why, we need to understand quantifiers.

Universal quantifier (∀): We use the symbol '∀' in FOL to express universal claims, like those we express in English with the phrases 'everything', 'each thing', 'all things', and 'anything'. It is always used in connection with a variable and so is said to be variable binding. We read '$\forall x$' as 'for every object x' or somewhat misleadingly 'for all x'. We translate the sentence 'Everything is at home' as

$\forall x\ \text{Home}(x)$

This says that every object x meets to following condition: x is at home. It says that everything whatsoever is at home. We rarely make unconditional claims like this. More often, we say something like 'All dogs are animals', translated

$\forall x(Dx \rightarrow Ax)$

This says that for every object x: if x is a dog then x is an animal. More naturally, for anything if it is a dog then it is an animal. To put it another way, take everything whatsoever, you'll find that it is either not a dog or it is an animal (remember that $p \rightarrow q \vdash \sim p \vee q$).

Existential quantifier (∃): We use the symbol '∃' in FOL to express existential claims, like those we can express in English with the phrases 'something', 'at least one thing', 'a', and 'an'. It is also always used in connection with a variable and so is said to be variable binding. We read '$\exists x$' as 'for some object x' or (misleadingly) 'for some x'. We translate 'Something is at home' as

$\exists x\ \text{Home}(x)$

This says that some object x meets the following condition: x is at home. It is more common however to express that something of a particular kind meets some condition, like 'Some dogs are huskies', translated

$\exists x(Dx \wedge Hx)$

This says that there is some object x that meets the complex condition: x is a dog and x is a husky. More colloquially, there is at least one dog that is a husky.

Free and bound variables: If constants are like names in a natural language, variables are like (certain uses of) pronouns. Paraphrase 'Every dog is an animal' as 'For everything, if it is a dog then it is an animal' translated to FOL as $\forall x(Dx \rightarrow Ax)$.

We all know by now how literal FOL is. It doesn't understand what is meant by a sentence like 'He has good handwriting' even if we would in context. ALL sentences of FOL that make use of variables must also make use of quantifiers in a specific way. We distinguish between *free variables* and *bound variables*.[2]

[2]It doesn't make sense to talk about free and bound constants.

Formula with a free variable: Fx (It (x) is F).

Sentence with a bound variable: $\forall x(Rxx)$ (Everything is identical with itself).

Let's be more precise. Provided predicates and constants of our vocabulary, an *atomic sentence* is the combination of predicates and constants such that the number of constants matches the arity of the predicate (e.g. If R is a 2-placed predicate and n and m are constants, then Rnm is an atomic sentence. If F is a one-placed predicate and n is a constant, then Fn is an atomic sentence.).

An *atomic formula* is either (a) an atomic sentence or (b) an atomic sentence where at least one occurrence of a constant is replaced by a variable (e.g. For one-placed predicate F, constant n, and variable x, Fn and Fx are both atomic formulas).[3]

A *free variable* is a variable that occurs in an atomic formula.

All free variables of atomic formula P are free in $\forall xP$, except for x, which is said to be *bound*.

A formula with no free variables is a *sentence* of FOL.

Consider ($\star$) 'Some student owns some dog and it is a husky'.

Correct translation of ($\star$): $\exists x(Sx \wedge \exists y(Dy \wedge Oxy \wedge Hy))$

Incorrect translations of ($\star$): $\exists x(Sx \wedge \exists y(Dy \wedge Oxy) \wedge Hy)$

Incorrect translations of ($\star$): $\exists x(Sx \wedge \exists y(Dy \wedge Oxy)) \wedge Hy$

WHY?

Translate the following into FOL:

(1) All P's are Q's
(2) Some P's are Q's
(3) No P's are Q's (this is NOT the same as Not all P's are Q's)
(4) Some P's are not Q's
(5) There is no beer in the fridge!
(6) Ben has better handwriting than anyone in the class
(7) Someone has better handwriting than Robert. Some of his letters look like others.
(8) Doruntina is a second-time student of logic
(9) Noah is home whenever Mitch is at the library.

[3]From the last two sections, we add the following to our recursive definition of wff:

(i) If P is a wff and x a variable, then $\forall xP$ is a wff, (it is a sentence if all variables are bound) and

(ii) If P is a wff and x a variable, then $\exists xP$ is a wff (it is a sentence if all variables are bound).

Translate the following into English:

(1)$\forall x \exists y (Likes(x, y))$

(2)$\exists y \forall x (Likes(x, y))$

Universal Elimination (UE): From $\forall x P$, infer $P[n/x]$ (replace every free occurrence of x with n) so long as n is a constant (including arbitrary names). Example:

1	(1)	$\forall x(Dx \to Ax)$	A
1	(2)	$Dn \to An$	1 UE

Universal Introduction (UI): From P, infer $\forall x P[x/a]$ (replace every occurrence of a single constant a with x), provided that a doesn't occur in the sentences of the dependency numbers of P. Example:

1	(1)	$\forall x(Dx)$	A
1	(2)	Da	1 UE
1	(3)	$Da \vee Aa$	2 $\vee$I
1	(4)	$\forall x(Dx \vee Ax)$	3 UI

Arguments (translate (if can), prove if valid, give counterexample if invalid):

(1) Bob is a square. All squares have four sides. $\vdash$ Bob has four sides.

(2) Shabazz is taller than Aniela. $\vdash$ Aniela is shorter than Shabazz.

(3) Some students smoke $\vdash$ Not all students smoke.

(4) $\forall x(Fx \to Gx), \forall x(Gx \to Hx) \vdash \forall x(Fx \to Hx)$

(5) If one thing is to the left of another, then the latter is to the right of the former. $\vdash$ Phil is not to the left of anything not to the right of him.