Phi 201: Precept 9

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Exercise 1. Show that $\emptyset \cup A = A$

Exercise 2. Show that $A \subseteq A$

Exercise 3. Show that $\{a\} = \{a, a\}$

Mathematical Induction: Mathematical induction is used to prove universal claims, where the elements in the domain of the universal quantification are defined inductively. An *inductive/recursive* definition consists of three parts:

1. a basis clause, which specifies the basic element of the definition;
2. one or more inductive clauses, which specify how to generate additional elements; and,
3. a final clause, which declares that all elements are either basic elements or generated by inductive clauses.

Example 1. One way to define the natural numbers is:

1. basic clause: $0 \in \mathbb{N}$
2. inductive clause: if $n \in \mathbb{N}$ then the successor of n, written as $s(n)$ or suggestively $n + 1$, is also in $\mathbb{N}$
3. final clause: all elements of $\mathbb{N}$ are in the intersection of sets satisfying 1 and 2.

Example 2. Let A be the set of propositional formulas. Here is one way to define A :

1. basic clause: propositional variables $p, q, ...$ are elements of A.
2. inductive clause: if $\phi \in A$, then $\neg\phi \in A$; if $\phi, \psi \in A$, then $\phi \wedge \psi$, $\phi \vee \psi$, and $\phi \rightarrow \psi$ are all in A.
3. final clause: every elements of A arises from a finite number of the previous steps.

To prove by mathematical induction that every element of $\mathbb{N}$ has a certain property P we do two things:

(Base case) Prove that the property holds for 0. That is, prove $P(0)$.[1]

(Inductive step) Prove that if the property holds of a number n, then it holds of the successor of n. That is, prove $P(n) \rightarrow P(n+1)$.[2]

Exercise 4. Prove that $0 + 1 + ... + n = \frac{n(n+1)}{2}$.

Exercise 5. Let A be a set of formulas defined as follows:

1. $p \in A$
2. If $\phi, \psi \in A$, then $\phi \wedge \psi \in A$ and $\phi \vee \psi \in A$
3. Every element of A arises from a finite number of the previous steps.

Show that $p \vdash \phi$ for any $\phi \in A$.

[1] More generally for sets other than the natural numbers, prove the property holds for the basic cases.

[2] More generally for sets other than the natural numbers, prove that if the property holds for some element(s) then they hold for all elements generated by the induction step.

Answer to 5 and hint:

Exercise 1 (Hints): First show that $\emptyset \cup A \subseteq A$. Start with "Let $a \in \emptyset \cup A$" and use the definitions to show that $a \in A$. End with, "Hence, $\emptyset \cup A \subseteq A$ since a is arbitrary." Then show $A \subseteq \emptyset \cup A$ in a symmetric way. You have to do both directions.

Exercise 5:

Proof. **Base Case**: $p \vdash p$

Induction step: Suppose $p \vdash \phi$ and $p \vdash \psi$. By conjunction introduction of the two hypotheses, $p \vdash \phi \wedge \psi$. By disjunction introduction on either of the hypotheses, $p \vdash \phi \vee \psi$.

Conclusion: Therefore, by mathematical induction $p \vdash \phi$ for any $\phi \in A$. □