

(1) $\vdash \forall x(Fx \vee \neg Fx)$

Pf. (1) $Fa \vee \neg Fa$ SI(s) LEM
 (2) $\forall x(Fx \vee \neg Fx)$ IVI

(2) $\forall x Fx \vdash \exists x Fx$

Pf. 1 (1) $\forall x Fx$ A
 1 (2) Fa IVE
 1 (3) $\exists x Fx$ ZEI

(3) $\exists x Fx \vdash \exists x \exists y (Fx \wedge Fy)$

Pf. 1 (1) $\exists x Fx$ A
 2 (2) Fa A (for EE)
 2 (3) $Fa \wedge Fa$ Z \wedge I
 2 (4) $\exists y (Fa \wedge Fy)$ ZEI
 2 (5) $\exists x \exists y (Fx \wedge Fy)$ \uparrow EI
 1 (6) $\exists x \exists y (Fx \wedge Fy)$ 1,2,5 EE

(4) $\vdash \exists x (\exists y Fy \rightarrow Fx)$

Pf. (1) $\exists y Fy \vee \neg \exists y Fy$ SI(s) LEM
 2 (2) $\exists y Fy$ A (for VE)
 3 (3) Fa A (for EE)
 3 (4) $\exists y Fy \rightarrow Fa$ 2,3 CP
 3 (5) $\exists x (\exists y Fy \rightarrow Fx)$ \uparrow EI
 2 (6) $\exists x (\exists y Fy \rightarrow Fx)$ 2,3,5 EE
 7 (7) $\neg \exists y Fy$ A (for VE)
 7 (8) $\exists y Fy \rightarrow Fa$ 7 SI(s) NP
 7 (9) $\exists x (\exists y Fy \rightarrow Fx)$ 8 EI
 (10) $\exists x (\exists y Fy \rightarrow Fx)$ 1,2,6,7,9 VE

(5) $\sim \exists x Fx \vdash \neg \forall x \sim Fx$

(Pf \vdash) 1 (1) $\sim \exists x Fx$ A
 2 (2) Fa A (for RA)
 2 (3) $\exists x Fx$ ZEI
 1,2 (4) $\sim \exists x Fx \wedge \exists x Fx$ 3 \wedge I
 1 (5) $\sim Fa$ 3,4 RA
 1 (6) $\forall x \sim Fx$ SUI

(Pf \neg) 1 (1) $\forall x \sim Fx$ A
 2 (2) $\exists x Fx$ A (for RA)
 3 (3) Fa A (for BE)
 1 (4) $\sim Fa$ IVE
 1,3 (5) $Fa \wedge \sim Fa$ 3,4 \wedge I
 1,3 (6) $\sim \exists x Fx$ 2,5 RA
 1,2 (7) $\sim \exists x Fx$ 2,3,6 EE
 1,2 (8) $\exists x Fx \wedge \sim \exists x Fx$ 2,7 \wedge I
 1 (9) $\sim \exists x Fx$ 2,8 RA

(6) $\sim \forall x Fx \vdash \neg \exists x \sim Fx$

(Pf \vdash) 1 (1) $\sim \forall x Fx$ A
 2 (2) $\sim Fa$ A (for CP)
 2 (3) $\exists x \sim Fx$ ZEI
 (4) $\sim Fa \rightarrow \exists x \sim Fx$ 2,3 CP
 (5) $\sim \exists x \sim Fx \rightarrow \sim \sim Fa$ 4 SI(s) (contrapositive)
 6 (6) $\sim \exists x \sim Fx$ A (for RA)
 6 (7) $\sim \sim Fa$ 5,6 MP
 6 (8) Fa 7 DN
 6 (9) $\forall x Fx$ 8 UI
 1,6 (10) $\forall x Fx \wedge \sim \forall x Fx$ 1,9 \wedge I
 1 (11) $\sim \sim \exists x \sim Fx$ 6,10 RA
 1 (12) $\exists x \sim Fx$ 11 DN

(Pf \neg) 1 (1) $\exists x \sim Fx$ A
 2 (2) $\forall x Fx$ A (for RA)
 2 (3) Fa ZVE
 \uparrow (4) $\neg Fa$ A (for EE)
 2,4 (5) $Fa \wedge \neg Fa$ 3,4 \wedge I
 4 (6) $\neg \forall x Fx$ 2,5 RA
 1 (7) $\neg \forall x Fx$ 1,4,6 EE